$\frac{\pi}{6}$, $\pi\pm\frac{\pi}{6}$, $\pm\frac{\pi}{3}$, $\pi\pm\frac{\pi}{3}$; or in *general*, $n\pi\pm\frac{\pi}{2}$, $2n\pi\pm\frac{\pi}{6}$, $(2n+1)\pi\pm\frac{\pi}{6}$,

$2n\pi\pm\frac{\pi}{3}$, $(2n+1)\pi\pm\frac{\pi}{3}$. [*J. B. Faught.*]

IV. From equation (2), $\cos\theta=0$. $\therefore$ $\theta=90^\circ, 270^\circ, \ldots.$ If $x=\cos^2\theta$, the second factor becomes $x^2-x=-\frac{3}{16}$, and $x_1=\frac{3}{4}$, $x_2=\frac{1}{4}$. $\therefore$ $\cos\theta=\frac{1}{2}\sqrt{3}$ and $\cos\theta=\frac{1}{2}$. $\therefore$ $\theta=30^\circ, 330^\circ\ldots,$ and $\theta=60^\circ, 300^\circ\ldots,$ for *particular* values. [*J. C. Corbin.*]

V. Contracting the last two terms of (1) into a *product of cosines*, by the formula $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$, we have $\cos\theta+2\cos4\theta\cos\theta=0$, or $\cos\theta(1+2\cos4\theta)=0$, $\therefore$ $\cos\theta=0$, and $\theta=\frac{\pi}{2}$ or 90°, or in general $\frac{1}{2}(2n\pm1)$; also, $1+2\cos4\theta=0$, $\therefore$ $\cos4\theta=-\frac{1}{2}$, and $\theta=30^\circ, 60^\circ$, or in general $\frac{1}{6}(3n\pm1)$. [*G. I. Hopkins, Otto Clayton.*]

VI. Equation $(1)=\dfrac{\cos3\theta\sin3\theta}{\sin\theta}=0$. $\therefore$ $\cos3\theta=0\ldots.(a)$, or, $\sin3\theta=0\ldots.(b)$. From (a), $\theta=\frac{1}{6}(2n+1)\pi$; from (b), $\theta=\frac{1}{3}n\pi$.
[*O. W. Anthony.*]

VII. $\cos3\theta=\cos\theta\cos2\theta-\sin\theta\sin2\theta$; $\cos5=\cos\theta\cos4\theta-\sin\theta\sin4\theta$. $\sin\theta=\sqrt{1-\cos^2\theta}$, $\sin2\theta=2\cos\theta\sqrt{1-\cos^2\theta}$; $\sin4\theta=4\cos\theta(2\cos^2\theta-1)\sqrt{1-\cos^2\theta}$, $\cos2\theta=2\theta=2\cos^2\theta-1$. Substituting, etc., the given equation becomes $\cos4\theta-\cos^2\theta=-\frac{3}{16}$; whence $\cos\theta=\frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$, giving $\theta=60^\circ$ or 30° for *particular* values. [*A. H. Bell.*]

NOTE.—The particular values given for this problem in Bowser's *Treatise on Trigonometry*, page 128, are $\frac{\pi}{2}$, $\frac{2}{3}\pi$.—*Editor.*

Also solved by *P. S. BERG, A. L FOOTE, F. P. MATZ, P. H. PHILBRICK, J. F. W. SCHEFFER C. D. SCHMITT. W. I. TAYLOR*, and *G. B M. ZEER.*

47. Proposed by LEONARD E. DICKSON, A. M., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove that $(-1)(-1)=+1$.

Solutions by the PROPOSER; G. B. M. ZEER, A. M , Ph. D., Professor of Mathematics in Inter State College, Texarkana, Texas; H. W. DRAUGHON, Olio, Mississippi; P. H. PHILBRICK, M. S., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Pineville, Louisiana; J H. GROVE, Professor of Mathematics in Howard Payne College, Brownwood, Texas; P. S. BERG, Apple Creek, Ohio; W. I. TAYLOR, Baldwin University, Berea, Ohio; Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. L. FOOTE, C. E , Middlebury, Connecticut; and F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

I. *Assuming* the distributive law to hold, $(-1)\{(+1)+(-1)\}$, or 0,

$=(-1)(+1)+(-1)(-1)$. Assuming the commutative law, $(-1)(+1)=(+1)(-1)=-1$. $\therefore -1+(-1)(-1)=0$, or $(-1)(-1)=+1$.

This proof was suggested by a longer one due to Professor D. A. Hull of Upper Canada College [*L. E. Dickson.*]

II. $(-1)(-1)$ means that -1 is to be taken subtractively one time. $\therefore 0-(-1)=+1$. $\therefore (-1)(-1)=+1$. [*G. B. M. Zerr.*]

III. $-1\times a=-a$. $-1\times(a-1)=-(a-1)=-a+1$. $\therefore -1\times[(a-1)-a]=-a+1-(-a)=-a+1+a=1$ [*P. H. Philbrick.*]

IV. $(-1)(-1)=(-1)(+1)-(-1)(+2)=-1-(-2)=-1+2=+1$.

[*H. W. Draughon.*]

V. Definition: $-n$ is the number which added to $n=0$. We know that $(-1)\times 1=-1$; suppose $(-1)\times(-1)=x$. Adding we get, $(-1)(1-1)=x-1$. But $(-1)(1-1)=0$. $\therefore x-1=0$. $\therefore x=+1$. $\therefore (-1)(-1)=+1$.

[*J. H. Grove.*]

VI. To multiply one number by another we do to the first what is done to unity to produce the second. [See Smith's *Algebras*, Van Velzer and Slichter's *Univ. Alg.*] $\therefore (-5)(-3)=(-5)(-1-1-1)=-(-5)-(-5)-(-5)=+5+5+5=15$. Similarly, $(-1)(-1)=(-1)(-1)=-(-1)=+1$.

[*P. S. Berg, F. P. Matz.*]

VII. According to Wood's *Elements of Algebra, 17th edition*, we have $(-5)(-3)=+15$. Here -3 is to be subtracted 5 times; that is, -15 is to be *subtracted*. Now, subtracting -15 is the same as adding $+15$. Therefore, we have to *add* $+15$. Similarly, $(-1)(-1)=+1$.

[*W. I. Taylor, F. P. Matz.*]

VIII. The case $(-a)(-b)=+ab$ is purely conventional and consequently an assumption, which, however, does not deprive the result of its great importance to algebraic operations. [*J. F. W. Scheffer.*]

IX. For illustration, $(a-b)\times(c-b)=(c-b)a-(c-b)b$, but $(c-b)a=ac-ab$ and $(c-b)b=cb-b.b$, and we have $(a-b)(c-b)=ac-ab-(cb-bb)$. Now as we are to take cb less $b.b$ from $ac-ab$, we first take cb and we have $ac-ab-cb$; which is too much by $b.b$; we therefore add $b.b$ and get $ac-ab-cb+b.b$, but $b.b$ is found from $(-b)(-b)=+b.b$. Take $b=1$, then $(-1)(-1)=+1$.

[*A. L. Foote.*]

X. Revolve the vector $(+a)$ about its origin A, through an angle of 180°, and it will become the vector $(-a)$, or will be *multiplied* by (-1). Making a equal to unity, then revolving the vector (-1) about its origin A,

through an angle of 180°, and it will become the vector $(+1)$, or will be multiplied by (-1); that is, $(-1)(-1)=+1$. [*F. P. Matz.*]

PROBLEMS.

56. Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

(a), How much can be paid for a bond, bearing 5% interest and having ten years to run, so as to realize 3% on the investment? [*C. E. Myers*]; (b), At what price must the government sell 5% $100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3% bonds at par, to run ten years, interest payable anually, provided the buyer can invest all interest received at 4% interest payable anually? [*J. H. D.*]

57. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2 \dots\dots & a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2 \dots\dots & a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2 \dots\dots & a_3^2 \\ \dots & \dots & \dots & \dots \\ a_n^2 & a_n^2 & a_n^2 \dots\dots & s-a_n^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1 \dots\dots & a_1 \\ a_2 & s-a_2 & a_2 \dots\dots & a_2 \\ a_3 & a_3 & s-a_3 \dots\dots & a_3 \\ \dots & \dots & \dots & \dots \\ a_n & a_n & a_n \dots\dots & s-a_n \end{vmatrix}$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by B. F. BURLESON, Oneida Castle, New York.

Determine the radius of a circle circumscribing three tangent circles of a radii $a=15$, $b=17$, and $c=19$.

I. Solution by the PROPOSER; J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. H. BELL, Hillsboro, Illinois; and F. P. MATZ, M. Sc., Ph. D., Mechanicsburg, Pennsylvania.

The problem has two cases: first, when the three given circles are tangent internally to the required circle, as in the problem; and, second, when the required circle is tangent to them externally. But *one solution* involving the resolution of a quadratic equation, will give the answers to both cases. We give the figure for the *first* case only.